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Convergence Properties of Dikin's Affine Scaling Algorithm for Nonconvex Quadratic Minimization[1]

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Abstract. We study convergence properties of Dikin's affine scaling algorithm applied to nonconvex quadratic minimization. First, we show that the objective function value either diverges or converges Q-linearly to a limit. Using this result, we show that, in the case of box constraints, the iterates converge to a unique point satisfying first-order and weak second-order optimality conditions, assuming the objective function Hessian Q is rank dominant with respect to the principal submatrices that are maximally positive semidefinite. Such Q include matrices that are positive semidefinite or negative semidefinite or nondegenerate or have negative diagonals. Preliminary numerical experience is reported.

Key words: Affine-scaling algorithm, Hoffman's error bound, Linear convergence, Nonconvex quadratic minimization, Trust region subproblem

1. Introduction

We consider the nonconvex quadratic program (QP):

$$\min \quad f(x) \doteq \frac{1}{2}x^T Qx + c^T x \quad \text{subject to} \quad Ax = b,\ 0 \leqslant x \leqslant u, \tag{1}$$

where $Q \in \Re^{n\times n}$ is symmetric, $A \in \Re^{m\times n}$, $b \in \Re^m$, $c \in \Re^n$, and $u \in [0,\infty]^n$. We allow $m = 0$, in which case (1) has only bound constraints. The case of $Q = 0$ corresponds to linear program (LP). We assume that (1) has a feasible interior solution x, i.e., $Ax = b$ and $0 < x < u$.

An important class of algorithms for solving (1) is the affine-scaling (AS) algorithm, originally proposed by Dikin (1967, 1974, 1988) and rediscovered by Barnes (1986) and Vanderbei et al. (1986) for LP and by Ye (1989) and Ye and Tse (1989) for QP as simplification of Karmarkar's algorithm (1984). In the AS algorithm, starting with a feasible interior solution x^0, it generates x^{k+1} from x^k, for $k = 0, 1, \ldots,$ by

$$x^{k+1} \doteq x^k + t_k d^k, \tag{2}$$

where d^k is a global optimal solution of the trust-region subproblem:

$$\min \quad \frac{1}{2}d^T Qd + \nabla f(x^k)^T d \quad \text{subject to} \quad Ad = 0, \ \|(S^k)^{-1}d\| \leqslant \beta, \tag{3}$$

with $S^k \doteq \text{diag}[s_j^k]_{j=1}^n$, $s_j^k \doteq \min\{x_j^k, u_j - x_j^k\}$, and

$$t_k \doteq \arg\min \left\{ f(x^k + td^k) : 1 \leqslant t \leqslant \min \left\{ \bar{t}, \beta \min_{j:d_j^k > 0} \frac{u_j - x_j^k}{d_j^k}, \beta \min_{j:d_j^k < 0} \frac{x_j^k}{-d_j^k} \right\} \right\}. \tag{4}$$

Here $\beta \in (0, 1)$ and $\bar{t} \in [1, \infty)$ are fixed constants. Notice that t_k is well defined and $t_k = 1$ if $\bar{t} = 1$. A related stepsize rule is given by Bonnans and Bouhtou (1995). The subproblem (3) has been much studied and is known to be efficiently solvable; see Conn, Gould, and Toint (2000) and Ye (1992). It is readily seen from $\beta < 1$ that $Ax^k = b$ and $0 < x^k < u$ for all k. Hence $s_j^k > 0$ for all j and k.

There have been extensive studies of the global convergence and local linear convergence of the AS algorithm (2)–(4) for LP (Dikin, 1974; Barnes, 1986; Tsuchiya, 1991, 1992; Tseng and Luo, 1992; Monteiro et al., 1993; Saigal, 1996; Dikin and Roos, 1997), for convex QP (Ye and Tse, 1989; Sun, 1993; Bonnans and Bouhtou, 1995; Monteiro and Tsuchiya, 1998), and nonconvex QP (Ye, 1992; Bonnans and Bouhtou, 1995; Tseng and Ye, 2000). This algorithm is a feasible method in that it needs to be initialized with a feasible interior solution. It is a short-step method when we set $\bar{t} = 1$. It is a second-order method in that the objective function of (3) has a quadratic term. There have also been studies of infeasible method for LP by Muramatsu and Tsuchiya (1996), and long-step methods for LP, corresponding to setting $\bar{t} = \infty$; see Castillo and Barnes (2000), Dikin (1991), Gonzaga (1990), Mascarenhas (1997), Terlaky and Tsuchiya (1999), Tsuchiya and Muramatsu (1995), Tsuchiya and Monteiro (1996), Vanderbei and Lagarias (1990), Vanderbei and Hall (1993). The stepsize rule (4) mediates between short-step and long-step methods. Extensions to linearly constrained convex minimization (Gonzaga and Carlos, 1990; Sun, 1996) and nonconvex minimization (Bonnans and Pola, 1997; Monteiro and Wang, 1998; Conn at al., 2000), have also been studied, as have variants for LP (Monma and Morton, 1987; Vanderbei, 1989).

In the convex case $Q \succeq 0$, global convergence of the AS algorithm (2)–(4) with $\bar{t} = 1$ has been fully analyzed by Monteiro and Tsuchiya (1998), using results from Dikin and Roos (1997), Sun (1993), Tsuchiya and Muramatsu (1995). In particular, they established convergence of $\{x^k\}$ to a global optimal solution x^* and convergence of $\{p^k\}$ to the analytic center of a face of the optimal dual solution set associated with $\nabla f(x^*)$, assuming the optimal solution set of QP is nonempty and bounded; see Theorem 4.15 of Monteiro and Tsuchiya (1998). Our analysis will focus on the global convergence and local linear convergence of the AS algorithm for nonconvex QP. Previous analyses by Bonnans and Bouhtou (1995) and Ye (1992) require certain primal and dual nondegeneracy assumptions on the problem. Our analysis will not require such assumptions. In particular, we show that $\{f(x^k)\}$ converges linearly to its limit without any nondegeneracy assumption (see Theorem 1). Using this result, we show that, in the case of box constraints and Q being rank dominant with respect to its maximally positive-semidefinite principal submatrices, $\{x^k\}$ converges globally to a feasible solution satisfying first- and weak second-order optimality conditions for (1)–(see Theorem 2). The nonconvex case is considerably more difficult to analyze than the convex case. For example, convergence to a non-local-minimum is possible in the nonconvex case; see Tseng and Ye (2000).

In studying the AS algorithm (2)–(4), it is sometimes assumed for simplicity that $u_i = \infty$ for all i; see, e.g., Monteiro and Tsuchiya (1998) and Ye (1992). While (1) can be transformed into this special case by introducing additional variables and equality constraints, the transformed problem has greater size and the AS algorithm applied to the transformed problem is *not* equivalent to the AS algorithm applied to the original problem (1). For this reason we do not assume $u_i = \infty$ for all i and we work directly with finite upper bounds; also see Section 4 of Bonnans and Bouhtou (1995).

Throughout, points in $\Re^n$ are viewed as column vectors, superscript T denotes tranpose, and $\|\cdot\|$, $\|\cdot\|_1$, $\|\cdot\|_\infty$ denote the 2-norm, 1-norm, ∞-norm, respectively. For any symmetric matrix $B \in \Re^{n\times n}$, we write $B \succeq 0$ (respectively, $B \succ 0$) to mean B is positive semidefinite (respectively, positive definite). For any $B \in \Re^{m\times n}$, B_{ij} and B_j denote, respectively, the (i, j)th entry and the jth column of B. For any $I \subseteq \{1, \ldots, m\}$ and $J \subseteq \{1, \ldots, n\}$, B_{IJ} denotes the submatrix of B obtained by removing rows $i \notin I$ and columns $j \notin J$. For any $x \in \Re^n$, we denote by x_j the jth component of x and, for any $J \subseteq \{1, \ldots, n\}$, by x_J the subvector of x obtained by removing x_j, $j \notin J$. For any $x \in \Re^n$ and $y \in \Re^m$, we write (x, y) to mean $\begin{bmatrix} x \\ y \end{bmatrix}$. Also, $\doteq$ means "define".

2. Basic Properties

In this section we derive some basic properties of the AS algorithm that will be used to analyze its convergence. It is well known that, by defining

$$Q^k \doteq S^k Q S^k, \qquad A^k \doteq A S^k, \qquad c^k \doteq S^k \nabla f(x^k),$$

the subproblem (3) can be written equivalently as

$$\min \quad \frac{1}{2} w^T Q^k w + (c^k)^T w \quad \text{subject to} \quad A^k w = 0, \ \|w\| \leqslant \beta.$$

Let $B^k \in \Re^{n\times \ell}$ be a matrix whose columns form an orthonormal basis for $\mathrm{Null}(A^k)$. Then $A^k w = 0$ if and only if $w = B^k v$ for some $v \in \Re^\ell$, so the above subproblem reduces to the unconstrained subproblem; see page 293 of Ye (1992):

$$\min \quad \frac{1}{2}v^T(B^k)^T Q^k B^k v + (c^k)^T B^k v \quad \text{subject to} \quad \|v\| \leqslant \beta.$$

By using a well-known optimality condition for this unconstrained subproblem, e.g., Conn, Gould, and Toint (2000), Ye (1992), it can be seen that d^k, together with some Lagrange multipliers $\lambda_k \geqslant 0$ and $p^{k+1} \in \Re^m$, satisfies the following necessary and sufficient optimality condition for (3):

$$\begin{aligned} (Q + \lambda_k (S^k)^{-2})d^k + \nabla f(x^k) - A^T p^{k+1} &= 0, \\ (B^k)^T Q^k B^k + \lambda_k I &\succeq 0, \\ \lambda_k(\|(S^k)^{-1}d^k\| - \beta) &= 0. \end{aligned} \tag{5}$$

The second relation says that $Q + \lambda_k(S^k)^{-2}$ is positive semidefinite over $\mathrm{Null}(A)$. Letting

$$\tilde{r}^k \doteq \nabla f(x^k) - A^T p^{k+1},$$

it can be shown using (5) that d^k is a global minimum of the Lagrangian

$$\frac{1}{2}d^T(Q + \lambda_k(S^k)^{-2})d + (\tilde{r}^k)^T d$$

over $\mathrm{Null}(A)$. Thus

$$\frac{1}{2}(d^k)^T(Q + \lambda_k(S^k)^{-2})d^k + (\tilde{r}^k)^T d^k \leqslant 0$$

or, equivalently,

$$\frac{1}{2}(d^k)^T Q d^k + (\tilde{r}^k)^T d^k \leqslant -\frac{1}{2}\lambda_k \|(S^k)^{-1}d^k\|^2.$$

This implies

$$\begin{aligned} f(x^{k+1}) - f(x^k) &\leqslant f(x^k + d^k) - f(x^k) \\ &= \frac{1}{2}(d^k)^T Q d^k + \nabla f(x^k)^T d^k \\ &= \frac{1}{2}(d^k)^T Q d^k + (\tilde{r}^k)^T d^k \\ &\leqslant -\frac{1}{2}\lambda_k \|(S^k)^{-1}d^k\|^2, \end{aligned} \tag{6}$$

where the first inequality uses (2) and (4); the last equality also uses $Ad^k = 0$. Also, denoting

$$r^{k+1} \doteq Qd^k + \tilde{r}^k = \nabla f(x^{k+1}) - A^T p^{k+1},$$

we obtain from (5) that

$$\lambda_k (S^k)^{-2} d^k + r^{k+1} = 0. \tag{7}$$

Using (5)–(7), we have the following descent lemma for the AS algorithm.

LEMMA 1 *If* $\lambda_k = 0$, *then* x^{k+1} *is a global optimal solution of* (1). *If* $\lambda_k > 0$, *then*

$$f(x^{k+1}) - f(x^k) \leqslant -\frac{1}{2}\beta\|S^k r^{k+1}\|. \tag{8}$$

Proof. If $\lambda_k = 0$, then (7) implies $r^{k+1} = 0$. Also, by (5), Q is positive semidefinite over $\text{Null}(A)$, so, for any feasible solution x of (1), we have from $A(x - x^{k+1}) = 0$ that

$$f(x) - f(x^{k+1}) = (r^{k+1})^T(x - x^{k+1}) + \frac{1}{2}(x - x^{k+1})^T Q(x - x^{k+1}) \geqslant 0.$$

Thus x^{k+1} is a global optimal solution of (1).

If $\lambda_k > 0$, then (5) implies $\|(S^k)^{-1}d^k\| = \beta$ or, using (7), $\|S^k r^{k+1}\|/\lambda_k = \beta$. This together with (6) yields (8).

The next lemma shows that S^k changes gradually with k.

LEMMA 2 *For* $k = 0, 1, \ldots,$ *we have*

$$s_j^{k+1} \leqslant (1 + t_k\beta)s_j^k, \quad j = 1, \ldots, n.$$

Proof. Fix any $k \in \{0, 1, \ldots\}$. For each j with $s_j^k = x_j^k$, we have $x_j^k \leqslant u_j/2$ and $|d_j^k|/x_j^k \leqslant \beta$, so that

$$x_j^{k+1} = x_j^k + t_k d_j^k \leqslant (1 + t_k\beta)x_j^k.$$

Thus, if $s_j^{k+1} = x_j^{k+1}$, this implies $s_j^{k+1} \leqslant (1 + t_k\beta)s_j^k$. If $s_j^{k+1} = u_j - x_j^{k+1}$, this implies $u_j/2 \leqslant x_j^{k+1}$, so that $s_j^{k+1} = u_j - x_j^{k+1} \leqslant u_j/2 \leqslant x_j^{k+1} \leqslant (1 + t_k\beta)x_j^k = (1 + t_k\beta)s_j^k$. For each j with $s_j^k = u_j - x_j^k$, we have $x_j^k \geqslant u_j/2$ and $|d_j^k|/(u_j - x_j^k) \leqslant \beta$, so that

$$u_j - x_j^{k+1} = (u_j - x_j^k) - t_k d_j^k \leqslant (1 + t_k\beta)(u_j - x_j^k).$$

Thus, if $s_j^{k+1} = u_j - x_j^{k+1}$, this implies $s_j^{k+1} \leqslant (1 + t_k\beta)s_j^k$. If $s_j^{k+1} = x_j^{k+1}$, this implies $u_j/2 \geqslant x_j^{k+1}$, so that $s_j^{k+1} = x_j^{k+1} \leqslant u_j/2 \leqslant u_j - x_j^{k+1} \leqslant (1 + t_k\beta)(u_j - x_j^k) = (1 + t_k\beta)s_j^k$.

3. Linear Convergence in Objective Value

In this section we show that $\{f(x^k)\}$ either diverges or converges linearly to a limit. This result extends Theorem 1 of Sun (1993) for the convex case $Q \succeq 0$, which in turn extends Theorem 1 of Tseng and Luo (1992) for the linear case $Q = 0$; also see Lemma 2.4 of Monteiro and Tsuchiya (1998) and Lemma 4.11 of Monteiro and Wang (1998) for the convex/concave case. The proof uses ideas from the proofs of Lemma 3.1 and Theorem 3.2 of Luo and Tseng (1992).

THEOREM 1 *Either* $\{f(x^k)\} \downarrow -\infty$ *or* $\{f(x^k)\}$ *converges Q-linearly to a limit* v, *i.e., there exist index* $\bar{k}$ *and* $\rho \in (0,1)$ *such that*

$$0 \leqslant f(x^{k+1}) - v \leqslant \rho(f(x^k) - v) \quad \forall k \geqslant \bar{k}.$$

Proof. Since $f(x^{k+1}) \leqslant f(x^k)$ for all k by (8), then either $\{f(x^k)\} \downarrow -\infty$ or else $\{f(x^k)\}$ converges to a limit v. We consider this latter case. Then, $\{f(x^{k+1}) - f(x^k)\} \to 0$. By Lemma 1, it suffices to consider the case of $\lambda^k > 0$ for all k. Denote

$$\eta^k \doteq S^k r^k.$$

Then, for each $k \geqslant 1$, Lemma 2 and S^k being diagonal imply

$$\|\eta^k\| \leqslant (1 + t_{k-1}\beta)\|S^{k-1} r^k\| \leqslant 2(1+\bar{t}\beta)(f(x^{k-1}) - f(x^k))/\beta \to 0 \quad \text{as } k \to \infty, \tag{9}$$

where the second inequality uses Lemma 1 and $t_k \leqslant \bar{t}$. For any pair of disjoint subsets I, J of $\{1, \ldots, n\}$, let

$$K_{I,J} \doteq \left\{ k \in \{0,1,\ldots\} : \begin{array}{r} s_j^k = x_j^k \leqslant \sqrt{|\eta_j^k|}\ \forall j \in I, \\ s_j^k = u_j - x_j^k \leqslant \sqrt{|\eta_j^k|}\ \forall j \in J, \\ |r_j^k| < \sqrt{|\eta_j^k|}\ \forall j \notin I \cup J \end{array} \right\}. \tag{10}$$

Since $s_j^k |r_j^k| = |\eta_j^k|$ so that either $s_j^k \leqslant \sqrt{|\eta_j^k|}$ or else $|r_j^k| < \sqrt{|\eta_j^k|}$, it follows that each $k \in \{0, 1, \ldots\}$ belongs to $K_{I,J}$ for some I, J. Since the number of pairs I, J is finite, there is at least one pair I, J such that $K_{I,J}$ is infinite.

Consider any I, J such that $K_{I,J}$ is infinite. Let $K \doteq \{1, \ldots, n\} \setminus (I \cup J)$. Let $L \doteq \{j \in \{1, \ldots, n\} : u_j < \infty\}$. Then, for all $k \in K_{I,J}$, the following linear system

$$\begin{array}{c} x_I = s_I^k, \quad x_J = u_J - s_J^k, \quad Q_j^T x - A_j^T p = -c_j + r_j^k \ \ \forall j \in K, \\ x \geqslant 0, \quad x_L \leqslant u_L, \quad Ax = b, \end{array} \tag{11}$$

has a solution, e.g., $(x, p) = (x^k, p^k)$. Now, (10) implies

$$\|(s_I^k, s_J^k, r_K^k)\|^2 \leqslant \|\eta^k\|_1 \quad \forall k \in K_{I,J}, \tag{12}$$

so (9) yields $\{(s_I^k, s_J^k, r_K^k)\}_{k \in K_{I,J}} \to 0$. Thus, the right-hand side of (11) is uniformly bounded for $k \in K_{I,J}$, so an error bound of Hoffman (1952) implies that (11) has a solution (y^k, q^k) that is bounded for $k \in K_{I,J}$. Since $\{(s_I^k, s_J^k, r_K^k)\}_{k \in K_{I,J}} \to 0$, any cluster point (y, q) of $\{(y^k, q^k)\}_{k \in K_{I,J}}$ satisfies

$$\begin{array}{c} y_I = 0, \quad y_J = u_J, \quad Q_j^T y - A_j^T q = -c_j \ \ \forall j \in K, \\ y \geqslant 0, \quad y_L \leqslant u_L, \quad Ay = b. \end{array} \tag{13}$$

Thus, this linear system has a solution. Let $\Sigma_{I,J}$ denote the set of solutions for (13). Since (x^k, p^k) is a solution of (11), an error bound of Hoffman (1952) implies there exists $(\bar{x}^k, \bar{p}^k) \in \Sigma_{I,J}$ satisfying

$$\|(\bar{x}^k, \bar{p}^k) - (x^k, p^k)\| \leqslant C_1 \|(s_I^k, s_J^k, r_K^k)\| \quad \forall k \in K_{I,J}, \tag{14}$$

where C_1 is a constant depending on Q, A, I, J only.

We claim that f is constant on each $\Sigma_{I,J}$. If (y, q) and (y', q') both belong to $\Sigma_{I,J}$, then (13) yields

$$\begin{aligned} f(y') - f(y) &= \frac{1}{2}(y' - y)^T Q(y' - y) + (Qy + c)^T (y' - y) \\ &= \frac{1}{2}(y' - y)^T Q(y' - y) + (Qy + c - A^T q)^T (y' - y) \\ &= \frac{1}{2}(y' - y)^T Q(y' - y), \end{aligned}$$

where the second equality uses $A(y' - y) = 0$ and third equality uses the $y'_j = y_j$ for all $j \in I \cup J$ and $Q_j^T y - A_j^T q = -c_j$ for all $j \notin I \cup J$. A symmetric argument yields

$$f(y) - f(y') = \frac{1}{2}(y - y')^T Q(y - y').$$

Combining the above two equalities yields $f(y') = f(y)$.

For any $k \in K_{I,J}$, we have

$$\begin{aligned} &(Q\bar{x}^k + c)^T (x^k - \bar{x}^k) \\ &= (Q\bar{x}^k + c - A^T \bar{p}^k)^T (x^k - \bar{x}^k) \\ &= \sum_{j \in I} (Q_j^T \bar{x}^k + c_j - A_j^T \bar{p}^k) x_j^k + \sum_{j \in J} (Q_j^T \bar{x}^k + c_j - A_j^T \bar{p}^k)(x_j^k - u_j) \\ &= \sum_{j \in I} (Q_j^T (\bar{x}^k - x^k) - A_j^T (\bar{p}^k - p^k) + r_j^k) s_j^k - \sum_{j \in J} (Q_j^T (\bar{x}^k - x^k) \\ &\quad - A_j^T (\bar{p}^k - p^k) + r_j^k) s_j^k, \end{aligned}$$

where the first equality uses $A(x^k - \bar{x}^k) = 0$, and the second equality uses $(\bar{x}^k, \bar{p}^k) \in \Sigma_{I,J}$. This together with

$$f(x^k) - f(\bar{x}^k) = \frac{1}{2}(x^k - \bar{x}^k)^T Q(x^k - \bar{x}^k) + (Q\bar{x}^k + c)^T (x^k - \bar{x}^k)$$

and the definition of η^k yield

$$
\begin{aligned}
&|f(x^k) - f(\bar{x}^k)| \\
&= \left| \frac{1}{2}(x^k - \bar{x}^k)^T Q(x^k - \bar{x}^k) + \left(\sum_{j\in I} (Q_j^T(\bar{x}^k - x^k) - A_j^T(\bar{p}^k - p^k))s_j^k + \eta_j^k \right) \right. \\
&\left. - \left(\sum_{j\in J} (Q_j^T(\bar{x}^k - x^k) - A_j^T(\bar{p}^k - p^k))s_j^k + \eta_j^k \right) \right| \\
&\leqslant \frac{1}{2}|(x^k - \bar{x}^k)^T Q(x^k - \bar{x}^k)| + \sum_{j\in I\cup J} \left(\left|(Q_j^T(\bar{x}^k - x^k) - A_j^T(\bar{p}^k - p^k))s_j^k\right| + |\eta_j^k| \right) \\
&\leqslant C_2\|x^k - \bar{x}^k\|^2 + \sum_{j\in I\cup J} \left(\left\|(Q_j, A_j)\right\| \|(\bar{x}^k, \bar{p}^k) - (x^k, p^k)\| s_j^k + |\eta_j^k| \right) \\
&\leqslant C_2 C_1^2 \|\eta^k\|_1 + \sum_{j\in I\cup J} \left\|(Q_j, A_j)\right\| C_1 \|\eta^k\|_1^{1/2} s_j^k + \|\eta^k\|_1 \\
&\leqslant C_2 C_1^2 \|\eta^k\|_1 + C_1 \|\eta^k\|_1^{1/2} \sum_{j\in I\cup J} \left\|(Q_j, A_j)\right\| \sqrt{|\eta_j^k|} + \|\eta^k\|_1,
\end{aligned}
$$

where C_2 is a constant depending on Q only, the third inequality uses (14) and (12), the last inequality uses (10) and $k \in K_{I,J}$. It follows that

$$
|f(x^k) - f(\bar{x}^k)| \leqslant C_{I,J}\|\eta^k\| \quad \forall k \in K_{I,J}, \tag{15}
$$

where $C_{I,J}$ is a constant depending on Q, A, I, J only. Let C be the maximum of $C_{I,J}$ over all pairs I, J such that $K_{I,J}$ is infinite.

Since $\{f(x^k)\} \downarrow v$ and, by (9), $\{\eta^k\} \to 0$, it follows from (15) that $\{f(\bar{x}^k)\}_{k\in K_{I,J}} \to v$. Since $\bar{x}^k \in \Sigma_{I,J}$ for all $k \in K_{I,J}$ and f is constant on $\Sigma_{I,J}$, this implies $f(\bar{x}^k) = v$ for all $k \in K_{I,J}$. This together with (15) and $C_{I,J} \leqslant C$ yields

$$
\begin{aligned}
f(x^k) - v &= f(x^k) - f(\bar{x}^k) \\
&\leqslant C\|\eta^k\| \\
&\leqslant 2C(1 + \bar{t}\beta)(f(x^{k-1}) - f(x^k))/\beta,
\end{aligned}
$$

for all $k \in K_{I,J}$, where the last inequality uses (9). It follows that

$$
0 \leqslant f(x^k) - v \leqslant \frac{2C(1/\beta + \bar{t})}{1 + 2C(1/\beta + \bar{t})}(f(x^{k-1}) - v).
$$

This is true for all $k \in K_{I,J}$ and all pairs I, J such that $K_{I,J}$ is infinite. Since each $k \in \{0, 1, \dots\}$ belongs to $K_{I,J}$ for some I, J, then this is true for all k sufficiently large. □

4. Global Convergence: Box Constrained Case

In this section, we consider the important box-constrained case, i.e., $m = 0$. In this case, (1) is primal nondegenerate. We say that a principal submatrix Q_{JJ} of Q ($\emptyset \neq J \subseteq \{1, \dots, n\}$) is *maximally psd* if $Q_{JJ} \succeq 0$ and Q_{JJ} is not a principal submatrix of any other positive semidefinite principal submatrix of Q. We consider the following assumption on Q:

$$Q_{Ji} \in \text{Range}(Q_{JJ}) \ \forall i \notin J \quad \text{whenever } Q_{JJ} \text{ is maximally psd.} \tag{16}$$

Roughly speaking, (16) says that Q is rank dominant with respect to its maximally psd principal submatrices. It is readily seen that (16) is satisfied whenever

(a) $Q \succeq 0$ (since Q is the only maximally psd principal submatrix of Q)

or (b) $Q \preceq 0$ (since any positive semidefinite principal submatrix Q_{JJ} must be the zero matrix, in which case $Q_{Ji} = 0$ for all i)

or (c) Q is nondegenerate, i.e., every principal submatrix of Q is nonsingular

or (d) all diagonal entries of Q are negative (since Q has no positive semidefinite principal submatrix)

or (e) $Q_{JJ} \succ 0$ and $Q_{jj} < 0$ for all $j \notin J$, for some $J \subseteq \{1, \dots, n\}$.

In particular, the example of Tseng and Ye (2000) satisfies (16). Thus, the class of Q satisfying (16) is large.

By using Lemmas 1 and 2, we show below that, under assumption (16), $\|d^k\|^2$ is in the order of the decrease in the f-value.

LEMMA 3 *Assume $m = 0$ and Q satisfies* (16). *If $\{x^k\}$ is bounded, then either x^k is a global optimal solution of* (1) *for some k or else there exists constant $c > 0$ such that $\|d^k\| \leqslant c\sqrt{\alpha^k}$ for all $k = 1, 2, \dots$, where* $\alpha^k \doteq \max\{f(x^{k-1}) - f(x^k), f(x^k) - f(x^{k+1})\}$.

Proof. Assume $\{x^k\}$ is bounded and x^k is not a global optimal solution of (1) for all k. We will argue that $\|d^k\| = O(\sqrt{\alpha^k})$ by contradiction. Suppose there exists a subsequence $K \subseteq \{1, 2, \dots\}$ such that

$$\frac{\|d^k\|}{\sqrt{\alpha^k}} \to \infty \quad \text{as } k \to \infty, k \in K.$$

By passing to a subsequence if necessary, we can assume that, for each j, either $\{|d_j^k|/\sqrt{\alpha^k}\}_{k \in K} \to \infty$ or $\{|d_j^k|/\sqrt{\alpha^k}\}_{k \in K}$ is bounded. Let

$$\bar{J} \doteq \left\{ j \in \{1, \dots, n\} : \{|d_j^k|/\sqrt{\alpha^k}\}_{k \in K} \to \infty \right\}.$$

Since $|d_j^k|/s_j^k \leqslant \beta$ for all j, this implies that

$$\frac{s_j^k}{\sqrt{\alpha^k}} \to \infty \quad \text{as } k \to \infty, k \in K, \quad \forall j \in \bar{J}. \tag{17}$$

Then, for all $j \in \bar{J}$, we have from Lemma 2 and $t_{k-1} \leqslant \bar{t}$ and (8) that $s_j^k |r_j^k| \leqslant (1+\bar{t}\beta)s_j^{k-1}|r_j^k| \leqslant 2(1+\bar{t}\beta)\alpha^k/\beta$, so that

$$\frac{|r_j^k|}{\sqrt{\alpha^k}} \leqslant \frac{2(1+\bar{t}\beta)}{\beta}\frac{\sqrt{\alpha^k}}{s_j^k} \;\to\; 0 \quad \text{as } k \to \infty, k \in K. \tag{18}$$

Since $\{x^k\}$ is bounded and $\{f(x^k)\}\downarrow$, then $\{f(x^k)\}$ converges so $\{\alpha^k\} \to 0$. Since (18) holds for all $j \in \bar{J}$, this implies $\{r_{\bar{J}}^k\}_{k\in K} \to 0$.

We claim that $Q_{\bar{J}\bar{J}} \succeq 0$, i.e., there does not exist a $u \in \Re^n$ satisfying

$$u^T Qu < 0 \quad \text{and} \quad u_j = 0\,\forall j \notin \bar{J}.$$

If such u exists, then by normalizing u if necessary we can assume that $\|u\| = \beta$. Then, letting

$$\tilde{d}^k \doteq (\min_{j\in\bar{J}} s_j^k)u$$

for all $k \in K$, we have

$$\|(S^k)^{-1}\tilde{d}^k\| \leqslant \beta,$$

so $\tilde{d}^k$ is a feasible solution of (3). Thus, using (2) and (4), we have

$$\begin{aligned} f(x^{k+1}) - f(x^k) &\leqslant f(x^k + d^k) - f(x^k) \\ &= \frac{1}{2}(d^k)^T Qd^k + \nabla f(x^k)^T d^k \\ &\leqslant \frac{1}{2}(\tilde{d}^k)^T Q\tilde{d}^k + \nabla f(x^k)^T \tilde{d}^k \\ &= \frac{1}{2}(\tilde{d}^k)^T Q\tilde{d}^k + (r^k)^T \tilde{d}^k \\ &= \frac{1}{2}(\min_{j\in\bar{J}} s_j^k)^2 u^T Qu + (\min_{j\in\bar{J}} s_j^k)\sum_{j\in\bar{J}} r_j^k u_j, \end{aligned}$$

and it follows from (17) and (18) that

$$\frac{f(x^{k+1}) - f(x^k)}{(\min_{j\in\bar{J}} s_j^k)^2} \leqslant \frac{1}{2}u^T Qu + \sum_{j\in\bar{J}} \frac{r_j^k}{(\min_{j\in\bar{J}} s_j^k)} u_j \to \frac{1}{2}u^T Qu < 0 \quad \text{as } k \to \infty, k \in K.$$

Since $(\min_{j\in\bar{J}} s_j^k)^2/\alpha^k \to \infty$ as $k \to \infty$, $k \in K$, this implies

$$\frac{f(x^k) - f(x^{k+1})}{\alpha^k} \to \infty \quad \text{as } k \to \infty, k \in K,$$

a contradiction of the definition of α^k.

Since $Q_{\bar{J}\bar{J}} \succeq 0$, there exists $J \subseteq \{1,\ldots,n\}$ such that $\bar{J} \subseteq J$ and Q_{JJ} is maximally psd. Let $J^c \doteq \{1,\ldots,n\} \setminus J$ and $\bar{J}^c \doteq \{1,\ldots,n\} \setminus \bar{J}$. Then, by (16),

$$Q_{JJ} = C^T C \quad \text{and} \quad Q_{JJ^c} = C^T CE$$

for some suitable matrices C and E. Let $\bar{C}$ and $\tilde{C}$ denote the matrices comprising the columns of C corresponding to $\bar{J}$ and $J \setminus \bar{J}$, respectively. Thus, for any d with $d_{\bar{J}^c} = d^k_{\bar{J}^c}$, we have

$$\begin{aligned} d^T Qd &= (d_J)^T Q_{JJ} d_J + 2(d_J)^T Q_{JJ^c} d_{J^c} + (d_{J^c})^T Q_{J^cJ^c} d_{J^c} \\ &= \|Cd_J\|^2 + 2(Cd_J)^T(CEd_{J^c}) + (d_{J^c})^T Q_{J^cJ^c} d_{J^c} \\ &= \|Cd_J + CEd^k_{J^c}\|^2 - \|CEd^k_{J^c}\|^2 + (d^k_{J^c})^T Q_{J^cJ^c} d^k_{J^c} \\ &= \|Cd_J + CEd^k_{J^c}\|^2 + \gamma^k \\ &= \|\bar{C}d_{\bar{J}} + \tilde{C}d^k_{J\setminus\bar{J}} + CEd^k_{J^c}\|^2 + \gamma^k \\ &= \|\bar{C}d_{\bar{J}} + h^k\|^2 + \gamma^k, \end{aligned} \tag{19}$$

where $\gamma^k \doteq (d^k_{J^c})^T Q_{J^cJ^c} d^k_{J^c} - \|CEd^k_{J^c}\|^2$ and $h^k \doteq \tilde{C}d^k_{J\setminus\bar{J}} + CEd^k_{J^c}$. Since $|d^k_j|/\sqrt{\alpha^k}$ is bounded for all $k \in K$ and all $j \in \bar{J}^c = (J \setminus \bar{J}) \cup J^c$, then $\|h^k\|/\sqrt{\alpha^k}$ is bounded for all $k \in K$. Also,

$$\begin{aligned} r^k_{\bar{J}} &= \nabla f(x^k)_{\bar{J}} \\ &= Q_{\bar{J}J} x^k_J + Q_{\bar{J}J^c} x^k_{J^c} + c_{\bar{J}} \\ &= \bar{C}^T C x^k_J + \bar{C}^T CE x^k_{J^c} + c_{\bar{J}}. \end{aligned}$$

Since $\{r^k_{\bar{J}}\}_{k\in K} \to 0$, this implies in the limit

$$0 = \bar{C}^T C y_J + \bar{C}^T CE y_{J^c} + c_{\bar{J}},$$

where y is any cluster point of $\{x^k\}_{k\in K}$. Thus,

$$r^k_{\bar{J}} = \bar{C}^T C(x^k_J + Ex^k_{J^c} - y_J - Ey_{J^c}) = \bar{C}^T Ce^k, \tag{20}$$

where $e^k \doteq x^k_J + Ex^k_{J^c} - y_J - Ey_{J^c}$.

Then, for each $k \in K$, (19) yields

$$\min_{d:d_{\bar{J}^c} = d^k_{\bar{J}^c}} \frac{1}{2} d^T Qd + (r^k)^T d \tag{21}$$

$$\iff \min_{d_{\bar{J}}} \frac{1}{2}\|\bar{C}d_{\bar{J}} + h^k\|^2 + (r^k_{\bar{J}})^T d_{\bar{J}} \;=\; \min_{d_{\bar{J}}} \frac{1}{2}\|\bar{C}d_{\bar{J}} + h^k\|^2 + (Ce^k)^T \bar{C} d_{\bar{J}}$$

$$\iff \min_{d_{\bar{J}}} \frac{1}{2}\|\bar{C}d_{\bar{J}} + h^k + Ce^k\|^2.$$

This is an unconstrained convex quadratic minimization in $d_{\bar{J}}$. Since the objective function is bounded from below, it has an optimal solution. Moreover, the necessary and sufficient optimality condition is

$$\bar{C}^T \bar{C} d_{\bar{J}} + \bar{C}^T h^k + r^k_{\bar{J}} = 0.$$

Since $\|h^k\|/\sqrt{\alpha^k}$ is bounded for $k \in K$ and (18) holds for all $j \in \bar{J}$, the above linear system has a solution $\hat{d}^k_{\bar{J}}$ in the order of $\sqrt{\alpha^k}$. Then, defining $\hat{d}^k_{\bar{J}^c} \doteq d^k_{\bar{J}^c}$, we see that $\hat{d}^k$ is a global optimal solution of (21). Since d^k is a feasible solution of (21), this implies that

$$\frac{1}{2}(\hat{d}^k)^T Q \hat{d}^k + (r^k)^T \hat{d}^k \leqslant \frac{1}{2}(d^k)^T Q(d^k) + (r^k)^T d^k.$$

Also, the definition of $\bar{J}$ and $\|\hat{d}^k_{\bar{J}}\| = O(\sqrt{\alpha^k})$ for $k \in K$ imply

$$\|(S^k)^{-1}\hat{d}^k\|^2 - \|(S^k)^{-1}d^k\|^2 = \sum_{j \in \bar{J}} \left(\frac{\hat{d}^k_j}{s^k_j}\right)^2 - \left(\frac{d^k_j}{s^k_j}\right)^2 < 0$$

for all $k \in K$ sufficiently large. Then, $\|(S^k)^{-1}\hat{d}^k\| < \beta$, so $\hat{d}^k$ is a global optimal solution of (3). This implies (see (5) and Lemma 1) that $x^k + \hat{d}^k$ is a global optimal solution of (1) with $m = 0$. Then $f(x^{k+1}) \leqslant f(x^k + d^k) = f(x^k + \hat{d}^k)$, where the equality is due to both d^k and $\hat{d}^k$ being global optimal solutions of (3). Thus x^{k+1} is a global optimal solution of (1). This contradicts our assumption on $\{x^k\}$. □

Using Theorem 1 and Lemma 3, we show below that, under assumption (16), $\{x^k\}$ attains global convergence and local linear convergence to a point x satisfying first-order and weak second-order optimality. Recall that a feasible solution x of (1) with $m = 0$ satisfies 1st-order and weak 2nd-order optimality conditions if

$$Q_j^T x + c_j \begin{cases} \leqslant 0 & if x_j = 0 \\ \geqslant 0 & if x_j = u_j \\ = 0 & else \end{cases} \qquad \text{and} \qquad Q_{JJ} \succeq 0,$$

with $J \doteq \{j \in \{1, \ldots, n\} : 0 < x_j < u_j\}$; see Conn, Gould, and Toint (2000), Ye (1992).

THEOREM 2 *Assume* $m = 0$ *and* Q *satisfies* (16). *Then either* (i) $\{x^k\}$ *is not bounded or* (ii) x^k *is a global optimal solution of* (1) *for some* k *or* (iii) $\{x^k\}$ *converges Q-linearly to some limit point* x^∞ *satisfying first-order and weak second-order optimality conditions for* (1).

Proof. Suppose $\{x^k\}$ is bounded and x^k is not a global optimal solution of (1) for all k. Since $\{f(x^k)\}$ is non-increasing, then $\{f(x^k)\}$ converges. By Theorem

1, $\{f(x^k)\}$ converges at a R-linear rate. Then $\{\alpha^k\} \to 0$ Q-linearly, where α^k is defined as in Lemma 3. Thus $\{\sqrt{\alpha^k}\} \to 0$ Q-linearly and hence, by Lemma 3 and (2) and $1 \leqslant t_k \leqslant \bar{t}$, $\{\|x^{k+1} - x^k\|\} \to 0$ Q-linearly. Thus, $\{x^k\}$ converges Q-linearly to some limit point x^∞. Then $\{r^k\}$ also converges. The conclusion then follows from Theorem 2 of Ye (1992), slightly modified to take into account the upper bound constraints $x \leqslant u$. □

It is an open question whether the assumptions of $m = 0$ and (16) in Theorem 2 can be relaxed. Perhaps the (highly nontrivial) analysis of Monteiro and Tsuchiya (1998) for the convex case, together with Theorem 1 and ideas from the proof of Theorem 2, can be applied to tackle this question. This is a direction for future research.

5. Numerical Experience

For LP, the practical performance of the AS algorithm has been well studied, e.g., Monma and Morton (1987), although now a days predictor-corrector primal-dual interior point methods are preferred. For convex QP, some numerical experience with a version of the AS algorithm is reported in Bonnans and Bouhtou (1995). For nonconvex QP, we are unaware of any published computational study of the AS algorithm. To gain better understanding of this issue, we describe in this section a Matlab implementation of the AS algorithm (2)–(4) for the box-constrained case, and we report our preliminary numerical experience with it.

In our Matlab implementation, we set $\beta = 0.95$ and $\bar{t} = 10$. Each subproblem (3) is solved using the Moré-Sorensen method, as described in Section 7.3 of Conn et al., (2000), with a precision of 10^{-12}. The AS algorithm is initialized with $x^0 = u/2$ and terminates at iteration k when the residual $\|x^k - \max\{0, \min\{u, x^k - \nabla f(x^k)\}\}\|$ is below the threshold 10^{-7}. For test problems, we use the nonconvex (indefinite) box-constrained QP problems, with $n = 100$ and varying condition number (ncond) and number of negative eigenvalues (negeig) of Q, generated as described on page 392 of Moré and Toraldo (1989). Table 1 tabulates the performance of the AS algorithm (2)–(4) on these problems, showing the iteration count and the residual on termination. It can be seen that the performance of the AS algorithm is relatively insensitive to ncond and negeig, requiring number of iterations ranging from 21 to 64. In contrast, the active-set gradient-projection method of Moré and Toraldo seems to be more sensitive to ncond and negeig (see page 398 of Moré and Toraldo, 1989), requiring number of iterations ranging from 1 to 141, depending also on the starting point. Of course, caution must be exercised in comparing the methods since the work per iteration, the starting points, the termination criterion, and the quality of the generated (local) solutions are not the same.

Table 1. Iteration counts and residuals on termination of the AS algorithm on nonconvex Moré-Toraldo test problems

ncond	negeig			
	10	25	50	90
0	$35/1 \cdot 10^{-8}$	$45/3 \cdot 10^{-8}$	$42/1 \cdot 10^{-9}$	$55/4 \cdot 10^{-12}$
3	$61/7 \cdot 10^{-11}$	$56/1 \cdot 10^{-8}$	$43/4 \cdot 10^{-10}$	$52/2 \cdot 10^{-11}$
6	$42/6 \cdot 10^{-12}$	$59/1 \cdot 10^{-9}$	$60/1 \cdot 10^{-11}$	$21/4 \cdot 10^{-8}$
9	$64/4 \cdot 10^{-12}$	$55/1 \cdot 10^{-11}$	$62/1 \cdot 10^{-11}$	$21/3 \cdot 10^{-8}$
12	$61/2 \cdot 10^{-11}$	$62/5 \cdot 10^{-12}$	$52/4 \cdot 10^{-14}$	$57/2 \cdot 10^{-12}$

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